Analysis of Inflection Points for Planar Cubic Bézier Curve

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***Abstract*—After analyzing the curvature expression, the inflection points were given by the known planar cubic Bézier control polygon information. Based on that, we proposed an algorithm that can quickly and accurately obtain the shape characteristic points of curve. Experimental results show that the algorithm is rapid, accurate, and robust.**

I. Introduction

In practice application, it is necessary to accurately describe the shape characteristic of the parametric curve. In general, It mostly depends on the analysis of the Bernstein basis function and control polygon to describe the shape characteristic of Bézier curve [1]. When peoples need to divide some segmented convex curve from the shape characteristic points [2], the method of extracting the characteristic points based on curvature expression have to be considered.

In order to quickly and accurately obtain the shape characteristic points of planar cubic Bézier curve, according to the nu-uniform inflection points concept in some branch of mathematics [3][4][5], we classified inflection points into singular points and lingering points [6]. Then, some specific formula for calculating inflection points were given and an effective algorithm was proposed.

II. Curvature of Cubic Bézier Curve

Curvature is an important parameter of planar curve. In this section, formula for calculating inflection points that can be classified into singular points and lingering points would be derived by analyzing curvature expression.

The cubic Bézier curve is defined as follow:

$$\boldsymbol{C}(t) = \sum_{i=0}^{3} B_3^i(t)\boldsymbol{P_i} \qquad t \in [0,1] \tag{1}$$

Here, $\boldsymbol{P_i}$ are control points of Bézier curve, they constitute control polygon of Bézier curve. $B_3^i(t)$ are cubic Bernstein basis function shown as follow:

$$B_3^i(t) = \frac{3!}{i!(3-i)!} t^i (1-t)^{3-i} \qquad i = (0, \dots, 3) \tag{2}$$

When the curve lies on the XY plane, curvature can be expressed by equation (3):

$$\kappa(t) = \frac{|\boldsymbol{C}\,'(t) \times \boldsymbol{C}\,''(t)|}{(|\boldsymbol{C}\,'(t)|)^3} \tag{3}$$

Here, $\boldsymbol{C}\,'(t)$ is first derivative of $\boldsymbol{C}(t)$. It shows the velocity in physics and shows tangential direction in the geometry on the point $\boldsymbol{C}(t)$. The first derivative of cubic Bézier curve can be expressed by equation (4):

$$\begin{aligned}\boldsymbol{C}\,'(t) &= 3\sum_{i=0}^{2} B_2^i(t)(\boldsymbol{P_{i+1}} - \boldsymbol{P_i}) \\ &= 3(\boldsymbol{A_0} + 2t\boldsymbol{D_0} + t^2\boldsymbol{E_0})\end{aligned} \tag{4}$$

$\boldsymbol{C}\,''(t)$ is the second derivative of $\boldsymbol{C}(t)$. It shows the acceleration in physics and shows curved tendency in the geometry. The second derivative of cubic Bézier curve can be expressed by equation (5):

$$\begin{aligned}\boldsymbol{C}\,''(t) &= 6((1-t)\boldsymbol{D_0} + t\boldsymbol{D_1}) \\ &= 6(\boldsymbol{D_0} + t\boldsymbol{E_0})\end{aligned} \tag{5}$$

In equation (4) and (5):

$$\begin{aligned}\boldsymbol{A_i} &= \boldsymbol{P_{i+1}} - \boldsymbol{P_i} \qquad (\boldsymbol{i} = 0,1,2) \\ \boldsymbol{D_j} &= \boldsymbol{A_{j+1}} - \boldsymbol{A_j} \qquad (\boldsymbol{j} = 0,1) \\ \boldsymbol{E_k} &= \boldsymbol{D_{k+1}} - \boldsymbol{D_k} \qquad (\boldsymbol{k} = 0)\end{aligned} \tag{6}$$

Here, $\boldsymbol{A_i}$, $\boldsymbol{D_j}$ and $\boldsymbol{E_k}$ are called first-order, second-order and third-order control vector of cubic Bézier curve, individually. Fig.1 shows the case that $\boldsymbol{P_0}$ and $\boldsymbol{P_3}$ lying on the same side of $\boldsymbol{A_1}$.

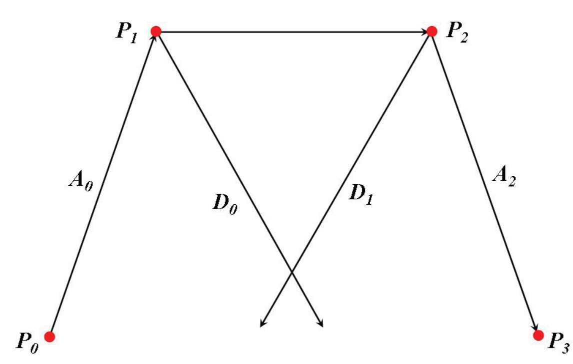

Fig. 1. Geometric meaning of control vector

Form the equation (4) and (5), $\boldsymbol{C}\,'(t)$ and $\boldsymbol{C}\,''(t)$ are continuous and derivable in the parametric interval $t \in [0,1]$.

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So $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t)$ shown in equation (3) is also continuous and derivable in the defined interval.

According to the theory of CAGD, $\boldsymbol{C}(t)$ are ***Inflection Points*** of curve when $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$. For vectors $\boldsymbol{C}'(t)$ and $\boldsymbol{C}''(t)$, there are only three possibility can lead to their cross product is 0:

- $\boldsymbol{C}'(t) \neq 0, \boldsymbol{C}''(t) \neq 0$ and $\boldsymbol{C}'(t) // \boldsymbol{C}''(t)$: Then $\kappa(t) = 0$ is tenable. In general, the points are called ***Lingering Points*** if they satisfied $\kappa(t) = 0$. So lingering points are special inflection points.
- $\boldsymbol{C}'(t) = 0$: equation (3) can not be calculated because $\kappa(t) = \frac{0}{0}$. In general, the points are called ***Singular Points*** if they satisfied $\boldsymbol{C}'(t) = 0$. So singular points are also special inflection points.
- $\boldsymbol{C}'(t) \neq 0$ and $\boldsymbol{C}''(t) = 0$: Because $\kappa(t) = 0$ is tenable, $\boldsymbol{C}(t)$ are lingering points.

According to the continuity, when $\boldsymbol{C}(t)$ is a lingering point, $\boldsymbol{C}''(t_-)$ and $\boldsymbol{C}''(t_+)$ should have the opposite direction for vector $\boldsymbol{C}'(t)$. While $\boldsymbol{C}(t)$ is a singular point, $\boldsymbol{C}'(t_-)$ and $\boldsymbol{C}'(t_+)$ should have the opposite direction for curve.

When use $\boldsymbol{A_{0x}}, \boldsymbol{A_{0y}}, \boldsymbol{D_{0x}}, \boldsymbol{D_{0y}}, \boldsymbol{E_{0x}}, \boldsymbol{E_{0y}}$ represent respectively the x and y components of vectors $\boldsymbol{A_0}$, $\boldsymbol{D_0}$ and $\boldsymbol{E_0}$, if:

$$\begin{aligned}
a &= \sqrt{\boldsymbol{A}_{0x}^2 + \boldsymbol{A}_{0y}^2} \\
d &= \sqrt{\boldsymbol{D}_{0x}^2 + \boldsymbol{D}_{0y}^2} \\
e &= \sqrt{\boldsymbol{E}_{0x}^2 + \boldsymbol{E}_{0y}^2} \\
f &= \boldsymbol{A_{0x}}\boldsymbol{D_{0x}} + \boldsymbol{A_{0y}}\boldsymbol{D_{0y}} \\
g &= \boldsymbol{A_{0x}}\boldsymbol{E_{0x}} + \boldsymbol{A_{0y}}\boldsymbol{E_{0y}} \\
h &= \boldsymbol{D_{0x}}\boldsymbol{E_{0x}} + \boldsymbol{D_{0y}}\boldsymbol{E_{0y}} \\
o &= \boldsymbol{A_{0x}}\boldsymbol{D_{0y}} - \boldsymbol{A_{0y}}\boldsymbol{D_{0x}} \\
p &= \boldsymbol{D_{0x}}\boldsymbol{E_{0y}} - \boldsymbol{D_{0y}}\boldsymbol{E_{0x}} \\
q &= \boldsymbol{A_{0x}}\boldsymbol{E_{0y}} - \boldsymbol{A_{0y}}\boldsymbol{E_{0x}}
\end{aligned} \tag{7}$$

Combined equations (4) and (5), necessary and sufficient conditions of inflection points exist can be obtain shown as equation (8):

$$pt^2 + qt + o = 0 \quad (t \in [0,1]) \tag{8}$$

The parameter $t \in [0,1]$ can be computed by equation (9):

$$t = \frac{-q \pm \sqrt{q^2 - 4po}}{2p} \quad (t \in [0,1]) \tag{9}$$

Especially:

- When $\boldsymbol{C}'(t) = 0$, necessary and sufficient conditions of singular points exist can be obtain shown as equation (10)

  $$e^2t^2 + 2ht + g = 2pt + q = 0 \quad (t \in [0,1]) \tag{10}$$

  Parameter $t \in [0,1]$ can be computed by equation (11):

  $$t = -\frac{q}{2p} = \frac{-h \pm \sqrt{h^2 - ge^2}}{e^2} \quad (t \in [0,1]) \tag{11}$$

  if $e = 0$, equations (10) and (11) can degenerate into equations (12) and (13):

  $$4d^2t^2 + 4ft + a^2 = 0 \quad (t \in [0,1]) \tag{12}$$

  $$t = \frac{-f \pm \sqrt{f^2 - a^2d^2}}{2d^2} \quad (t \in [0,1]) \tag{13}$$

- When $\boldsymbol{C}'(t) \neq 0$ and $\boldsymbol{C}''(t) = 0$, necessary and sufficient conditions of lingering points exist can be obtain shown as equation (14)

  $$e^2t^2 + 2ht + d^2 = 0 \quad (t \in [0,1]) \tag{14}$$

  Parameter $t \in [0,1]$ can be computed by equation (15):

  $$t = \frac{-h \pm \sqrt{h^2 - e^2d^2}}{e^2} \quad (t \in [0,1]) \tag{15}$$

III. Control Polygons of Cubic Bézier Curve

According the relative position relationship of the four control points for cubic Bézier curve, the control polygon will be one of the following three forms:

Form 1: $\boldsymbol{P_i}$ lying on the same line.

Form 2: $\boldsymbol{P_0}$, $\boldsymbol{P_3}$lying on both sides of $\boldsymbol{A_1}$.

Form 3: $\mathbf{P}_0\mathbf{P}_3$ lying on the same side of $\boldsymbol{A_1}$.

In order to find the shape characteristic points of the curve, all possibilities of the combination of curvature expression and control polygon should be discussed in detail.

Form 1: In the interval $t \in [0,1]$, $\boldsymbol{C}'(t)$ and $\boldsymbol{C}''(t)$ both have two possibilities of zero and non-zero.

- When $\boldsymbol{C}'(t) \neq 0$, if $\boldsymbol{C}''(t) \neq 0$, because it and $\boldsymbol{C}'(t)$ lying on the same line, $|\boldsymbol{C}'(t) \times \boldsymbol{C}''(t)| = 0$, that is $\kappa(t) = 0$ and each point on the curve is lingering point.
- When $\boldsymbol{C}'(t) = 0$, the singular point exist. Its parameter t can be obtained by equation (11) or (13).

Form 2: In the interval $t \in [0,1]$, if $\boldsymbol{C}'(t) \neq 0$ is always tenable, then singular points are not exist (***Theorem 1***). If the point which make $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ exist (***Theorem 2***), it must be lingering point of the curve. Now, we prove the two theorems.

Theorem 1: If $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on both sides of $\boldsymbol{A_1}$, $\boldsymbol{C}'(t) \neq 0$ is always tenable.

Prove: From equation (4), we can obtain equation (16):

$$\boldsymbol{C}'(t) = 3\left(\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_{i+1}} - \sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_i}\right) \tag{16}$$

that means $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_{i+1}}$ and $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_i}$ are quadratic Bézier curve defined by $\boldsymbol{P_1}$, $\boldsymbol{P_2}$, $\boldsymbol{P_3}$ and $\boldsymbol{P_0}$, $\boldsymbol{P_1}$, $\boldsymbol{P_2}$. According the convex hull property, $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_{i+1}}$ and $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_i}$ must lying on both sides of $\boldsymbol{A_1}$ (see Fig.2). From equation (16):

- When $t = 0$, $\boldsymbol{C}'(0) = 3\boldsymbol{A_0} \neq 0$.
- When $t = 1$, $\boldsymbol{C}'(1) = 3\boldsymbol{A_2} \neq 0$.

- For the arbitrary parameter $t \in (0,1)$, the start point and the end point of the vector $\boldsymbol{C}'(t)$ must be lying on both sides of $\boldsymbol{A_1}$, so $\boldsymbol{C}'(t) \neq 0$ is tenable.

Theorem 1 is proved.

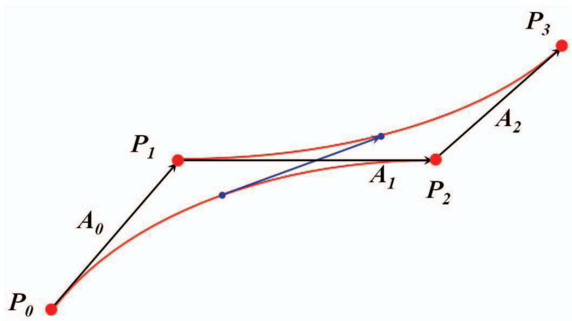

Fig. 2. Geometric meaning of $\boldsymbol{C}'(t)$

Theorem 2: If $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on both sides of $\boldsymbol{A_1}$, the point which make $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ ($t \in [0,1]$) must exist.

Prove: From equation (4), (5) and (6):

- When $t = 0$, $\boldsymbol{C}'(0) \times \boldsymbol{C}''(0) = \boldsymbol{A_0} \times \boldsymbol{A_1}$.
- When $t = 1$, $\boldsymbol{C}'(1) \times \boldsymbol{C}''(1) = \boldsymbol{A_1} \times \boldsymbol{A_2}$.

When $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on both sides od $\boldsymbol{A_1}$, $\boldsymbol{A_0} \times \boldsymbol{A_1}$ and $\boldsymbol{A_1} \times \boldsymbol{A_2}$ defined two vectors with opposite direction. According to the Mean Value Theorem, when $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t)$ is continuous and derivable, the point which make $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ ($t \in [0,1]$) must exist. ***Theorem 2*** is proved.

Form 3: In the interval $t \in [0,1]$, $\boldsymbol{D_0}$, $\boldsymbol{D_1}$ always point to the same side of $\boldsymbol{A_1}$, so $\boldsymbol{C}''(t) \neq 0$ is always tenable. However, the case $\boldsymbol{C}'(t) = 0$ and $\boldsymbol{C}'(t) // \boldsymbol{C}''(t)$ may exist.

When inflection points (including lingering points and singular points) were regarded as the shape characteristic points of cubic Bézier curve, according to our discussed results, lingering points from Form 1 were excluded, rapid extraction algorithm of shape characteristic points can be obtained as follows:

Step 1: According the shape of the control polygon to judge whether it belongs to the Form 1. If it's Form 1, turn to Step 3, otherwise, implement Step 2.

Step 2: To implement Step 3 after calculating the parameters and coordinates of inflection points based on equation (8) and (9).

Step 3: First, according the equation (10) (11) or equation (12) (13) to calculate the parameters and coordinates of singular points. Second, classified the inflection points from Step 2 into singular points set and lingering points set. Then, implement Step 4.

Step 4: According the equation (14) (15) to calculate the parameters and coordinates of lingering points, Then combining them to the lingering points set obtained from Step 3.

Step 5: Finished algorithm.

IV. Experimental Results

From Fig.3 to Fig.7 shows all kinds of shapes characteristic points of cubic Bézier curve included the Form 2 and Form

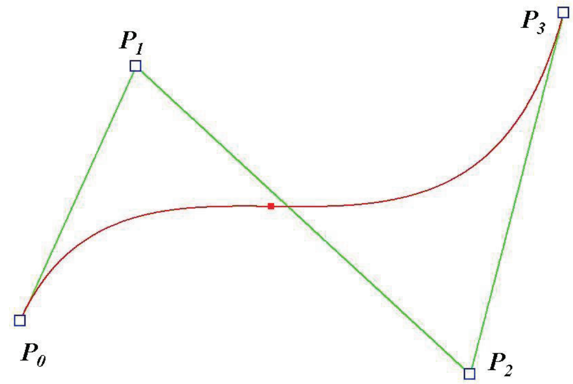

Fig. 3. $\boldsymbol{P_0 P_3}$ lying both sides of $\boldsymbol{A_1}$

3. Where the small solid rectangle represents lingering points and the large hollow rectangular represents singular points.

Fig.3 shown a case that $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying both sides of $\boldsymbol{A_1}$. In this case, there must be at least one lingering point. Fig.4 and 5 shown two cases that $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ did not cross with $\boldsymbol{A_2}$. In this two cases, there are two lingering points, respectively. In Fig.4, two lingering points have different parameter values and approximative coordinates values. Fig.6 and 7 shown two cases that $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ crossed with $\boldsymbol{A_2}$. In this two cases, there are two lingering points and one singular point, respectively. In Fig.6, two lingering points and one singular point have different parameter values and approximative coordinates values.

TABLE I shown the control points coordinates of cubic Bézier curves which be in from Fig.3 Fig.7, the classified of inflection points, the corresponding parameter values and coordinates values. The upper-left corner is the coordinate origin point, rightward is the X-axis' positive direction, downward is the Y-axis' positive direction. The coordinate values of the inflection points are approximative by substituted the corresponding parameters into the curve equation and taken the integer values of them.

V. Conclusion

The formula for calculating the inflection points which represent the shape characteristic of planar cubic Bézier curve were given based on the analysis of curvature expression. Combined the shape characteristic of the control polygons and classified inflection points, we proposed a algorithm which can quickly and correctly obtain shape characteristic points. The experimental results verify the correctness of the formula and excellent robustness of the algorithm.

Acknowledgment

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TABLE I
CURVE INFORMATION FROM FIG.3 TO FIG.7

Curve Information	Fig.3	Fig.4	Fig.5	Fig.6	Fig.7
Control points Coordinate Value	$\boldsymbol{P_0}$:(16, 467) $\boldsymbol{P_1}$:(185, 95) $\boldsymbol{P_2}$:(673, 545) $\boldsymbol{P_3}$:(810, 17)	$\boldsymbol{P_0}$:(859, 676) $\boldsymbol{P_1}$:(13, 422) $\boldsymbol{P_2}$:(781, 12) $\boldsymbol{P_3}$:(266, 425)	$\boldsymbol{P_0}$:(872, 686) $\boldsymbol{P_1}$:(11, 423) $\boldsymbol{P_2}$:(779, 13) $\boldsymbol{P_3}$:(220, 376)	$\boldsymbol{P_0}$:(819, 566) $\boldsymbol{P_1}$:(43, 18) $\boldsymbol{P_2}$:(826, 18) $\boldsymbol{P_3}$:(25, 533)	$\boldsymbol{P_0}$:(884, 574) $\boldsymbol{P_1}$:(135, 14) $\boldsymbol{P_2}$:(678, 14) $\boldsymbol{P_3}$:(14, 566)
Inflection Points Parameter Value	$t_0 = 0.456590$	$t_0 = 0.681076$ $t_1 = 0.705299$	$t_0 = 0.588071$ $t_1 = 0.886863$	$t_0 = 0.476169$ $t_1 = 0.539295$ $t_2 = 0.507732$	$t_0 = 0.320836$ $t_1 = 0.682291$ $t_2 = 0.501564$
Inflection Points Classification	t_0:Lingering	t_0:Lingering t_1:Lingering	t_0:Lingering t_1:Lingering	t_0:Lingering t_1:Lingering t_2:Singular	t_0:Lingering t_1:Lingering t_2:Singular
Inflection Points Coordinate Value	t_0:(383,300)	t_0:(461,249) t_1:(461,249)	t_0:(441,256) t_1:(363,281)	t_0:(431,152) t_1:(431,152) t_2:(431,152)	t_0:(360,207) t_1:(479,207) t_2:(416,152)

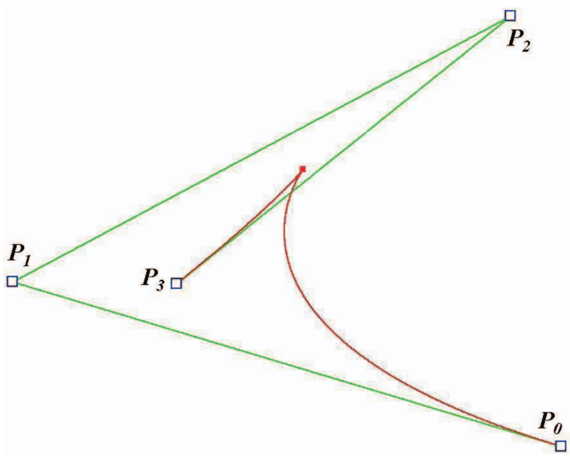

Fig. 4. $\boldsymbol{P_0P_3}$ lying the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ did not cross with $\boldsymbol{A_2}$, in this case there are two lingering points with approximative coordinates values

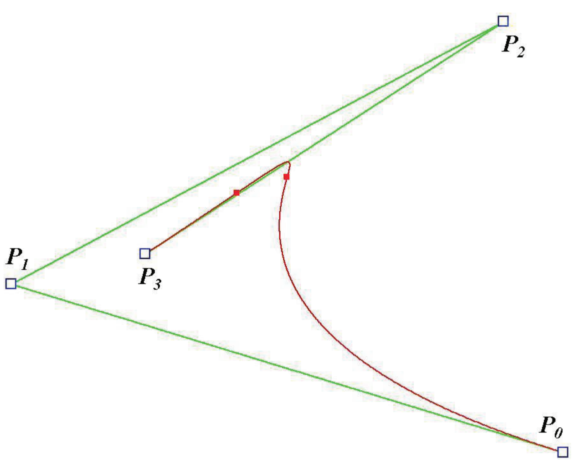

Fig. 5. $\boldsymbol{P_0P_3}$ lying the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ did not cross with $\boldsymbol{A_2}$

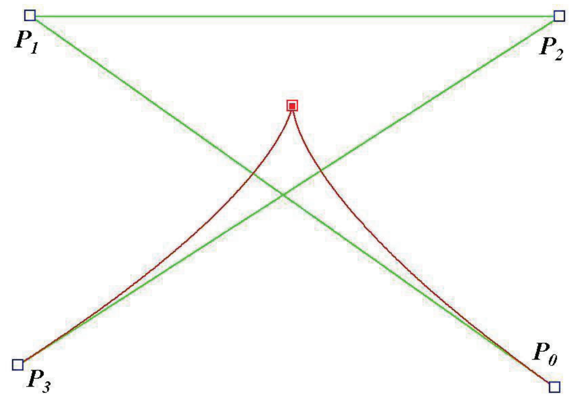

Fig. 6. $\boldsymbol{P_0P_3}$ lying the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ crossed with $\boldsymbol{A_2}$, there are two lingering and one singular point with approximative coordinates values

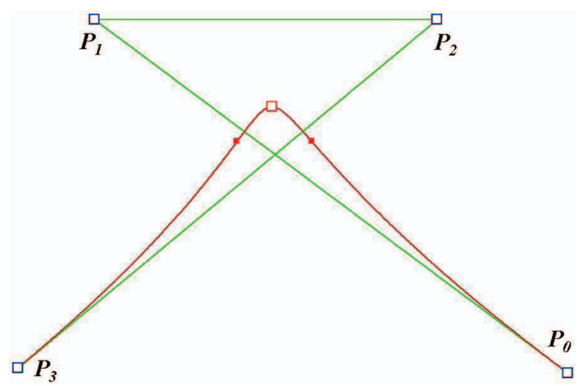

Fig. 7. $\boldsymbol{P_0P_3}$ lying the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ crossed with $\boldsymbol{A_2}$

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